IDENTICAL AND NONIDENTICAL RELATIONS. NONDEGENERATE AND DEGENERATE TRANSFORMATIONS

(Properties of skew-symmetric differential forms)

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Identical relations occur in various branches of mathematics and mathematical physics. The Cauchy-Riemann relations, characteristical and canonical relations, the Bianchi identities and others are examples of identical relations. It can be shown that all these relations express either the conditions of closure of exterior (skew-symmetric) differential forms and corresponding dual forms or the properties of closed exterior forms. Since the closed differential forms are invariant under all transformations, which conserve the differential (these are gauge transformations: unitary, canonical, gradient and others), from this it follows that identical relations are a mathematical representation of relevant invariant and covariant objects, which are of great functional and utilitarian importance.

The theory of exterior differential forms, which describes invariant objects using identical relations, cannot answer the question of how do invariant objects appear and what does these objects generate?

The answer to this question can be obtained using the skew-symmetric differential forms, which possess the evolutionary properties. The mathematical apparatus of such evolutionary forms contains nonidentical relations, from which the identical relations corresponding to invariant objects are obtained with the help of degenerate transformations.

Due to such potentialities, the mathematical apparatus of skew-symmetrical differential forms enables one to describe discrete transitions, evolutionary processes and generation of various structures.

1 Identical relations and nondegenerate transformations

Identical relations lie at the basis of the mathematical apparatus of exterior differential forms. They reflect the properties of exterior forms. Below we present some properties of closed exterior differential forms, which are necessary for further presentation. (In more detail about skew-symmetric differential forms one can read in [1-7]).

Closed exterior differential forms

The exterior differential form of degree p (p-form on the differentiable manifold) can be written as [1,3]

$$\theta^p = \sum_{i_1 \dots i_p} a_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \quad 0 \leq p \leq n \tag{1.1}$$

Here $a_{i_1 \dots i_p}$ are the functions of the variables x^{i_1}, x^{i_2}, $\dots$, x^{i_p}, n is the dimension of space, $\wedge$ is the operator of exterior multiplication, dx^i, $dx^i \wedge dx^j$, $dx^i \wedge dx^j \wedge dx^k$, $\dots$ is the local basis, which satisfies the condition of exterior multiplication:

$$\begin{aligned} &dx^i \wedge dx^i = 0 \\ &dx^i \wedge dx^j = -dx^j \wedge dx^i \quad i \neq j \end{aligned} \tag{1.2}$$

The differential of the (exterior) form θ^p is expressed as

$$d\theta^p = \sum_{i_1 \dots i_p} da_{i_1 \dots i_p} dx^{i_1} dx^{i_2} \dots dx^{i_p} \tag{1.3}$$

and is the differential form of degree $(p+1)$.

[From here on the symbol $\sum$ can be omitted and it will be implied that a summation over double indices is performed. Besides, the symbol of exterior multiplication will be also omitted for the sake of presentation convenience].

Let us consider the exterior differential form of the first degree $\omega = a_i dx^i$. In this case the differential will be expressed as $d\omega = K_{ij} dx^i dx^j$, where $K_{ij} = (\partial a_j / \partial x^i - \partial a_i / \partial x^j)$ are components of the form commutator.

In this section we will consider the domains of Euclidean space or differentiable manifolds [2]. (Manifolds, on which the skew-symmetric differential forms may be defined, and the influence of the manifold properties on the differential forms will be discussed in more detail in section 2).

The exterior differential form of degree p (p-form on the differentiable manifold) is called a closed one if its differential is equal to zero:

$$d\theta^p = 0 \tag{1.4}$$

A differential of the form is a closed form. That is,

$$dd\omega = 0 \tag{1.5}$$

where ω is an arbitrary exterior form.

The form that is a differential of some other form:

$$\theta^p = d\theta^{p-1} \tag{1.6}$$

is called an *exact* form. Exact forms prove to be closed automatically [2]

$$d\theta^p = dd\theta^{p-1} = 0 \tag{1.7}$$

Here it is necessary to call attention to the following points. In the above presented formulas it was implicitly assumed that the differential operator d is a total one (that is, the operator d acts everywhere in the vicinity of the point considered locally), and therefore, it acts on the manifold of the initial dimension n. However, a differential may be internal. Such a differential acts on some structure with the dimension being less than that of the initial manifold. The structure, on which the exterior differential form may become a closed *inexact* form, is a pseudostructure with respect to its metric properties. {Cohomology, sections of cotangent bundles, the eikonal surfaces, the characteristical and potential surfaces, and so on may be regarded as examples of pseudostructures.}

If the form is closed on pseudostructure only, the closure condition is written as

$$d_\pi \theta^p = 0 \tag{1.8}$$

And the pseudostructure π is defined from the condition

$$d_\pi {}^*\theta^p = 0 \tag{1.9}$$

where ${}^*\theta^p$ is a dual form. (For the properties of dual forms see [5]).

The fundamental properties of exterior differential forms are connected with the fact that any closed form is a differential. The exact form is, by definition, a differential (see condition (1.6)). In this case the differential is total. The closed inexact form is a differential too. The closed inexact form is an interior (on pseudostructure) differential, that is

$$\theta^p_\pi = d_\pi \theta^{p-1} \tag{1.10}$$

And so, any closed form is a differential of the form of lower degree: the total one $\theta^p = d\theta^{p-1}$ if the form is exact, or the interior one $\theta^p = d_\pi \theta^{p-1}$ on pseudostructure if the form is inexact.

The closure of exterior differential forms result from the conjugacy of elements of exterior or dual forms.

The closure property of the exterior form means that any objects, namely, elements of the exterior form, components of elements, elements of the form differential, exterior and dual forms and others, turn out to be conjugated. A variety of objects of conjugacy leads to the fact that there is a large number of different types of closed exterior forms.

Since the conjugacy is a certain connection between two operators or mathematical objects, it is evident that, to express a conjugacy mathematically, it can be used relations. Just such relations constitute the basis of mathematical apparatus of the exterior differential forms. This is an identical relation.

Identical relations of exterior differential forms

The identical relations reflect the closure conditions of the differential forms, namely, vanishing the form differential (see formulas (1.4), (1.8), (1.9)) and

the conditions connecting the forms of consequent degrees (see formulas (1.6), (1.10)).

The importance of the identical relations for exterior differential forms is manifested by the fact that practically in all branches of physics, mechanics, thermodynamics one faces such identical relations. One can present the following examples:

a) the Poincare invariant $ds = -H\,dt + p_j\,dq_j$,

b) the second principle of thermodynamics $dS = (dE + p\,dV)/T$,

c) the vital force theorem in theoretical mechanics: $dT = X_i dx^i$ where X_i are the components of potential force, and $T = mV^2/2$ is the vital force,

d) the conditions on characteristics in the theory of differential equations, and so on.

The identical relations in differential forms express the fact that each closed exterior form is a differential of some exterior form (with the degree less by one). In general form such an identical relation can be written as

$$d_\pi \phi = \theta_\pi^p \tag{1.11}$$

In this relation the form in the right-hand side has to be a *closed* one. (As it will be shown below, the identical relations are satisfied only on pseudostructures).

In identical relation (1.11) in one side it stands the closed form and in other side does the differential of some differential form of the less by one degree, which is the closed form as well.

In addition to relations in the differential forms from the closure conditions of differential forms and the conditions connecting the forms of consequent degrees the identical relations of other types are obtained. The types of such relations are presented below.

1. *Integral identical relations.*

The formulas by Newton, Leibnitz, Green, the integral relations by Stokes, Gauss-Ostrogradskii are examples of integral identical relations.

2. *Tensor identical relations.*

From the relations that connect exterior forms of consequent degrees one can obtain the vector and tensor identical relations that connect the operators of the gradient, curl, divergence and so on.

From the closure conditions of exterior and dual forms one can obtain the identical relations such as the gauge relations in electromagnetic field theory, the tensor relations between connectednesses and their derivatives in gravitation (the symmetry of connectednesses with respect to lower indices, the Bianchi identity, the conditions imposed on the Christoffel symbols) and so on.

3. *Identical relations between derivatives.*

The identical relations between derivatives correspond to the closure conditions of exterior and dual forms. The examples of such relations are the above presented Cauchy-Riemann conditions in the theory of complex variables, the transversality condition in the calculus of variations, the canonical relations

in the Hamilton formalism, the thermodynamic relations between derivatives of thermodynamic functions [8], the condition which the derivative of implicit function is subject to, the eikonal relations [9] and so on.

Nondegenerate transformations

One of the fundamental methods in the theory of exterior differential forms is an application of *nondegenerate* transformations (below it will be said about *degenerate* transformations).

In the theory of exterior differential forms the nondegenerate transformations are those that conserve the differential. This is connected with the property of closed differential forms. Since a closed form is a differential (a total one, if the form is exact, or an interior one on pseudostructure, if the form is inexact), it is evident that the closed form turns out to be invariant under all transformations that conserve the differential.

The examples of nondegenerate transformations in the theory of exterior differential forms are unitary, tangent, canonical, gradient transformations.

To the nondegenerate transformations there are assigned closed forms of given degree. To the unitary transformations it is assigned (0-form), to the tangent and canonical transformations it is assigned (1-form), to the gradient transformations it is assigned (2-form) and so on. It should be noted that these transformations are *gauge transformations* for spinor, scalar, vector, tensor (3-form) fields.

The connection between nondegenerate transformations and closed exterior forms disclose an internal commonness of nondegenerate transformations: all these transformations are transformations that preserve a differential and nondegenerate transformations can be classified by a degree of corresponding closed differential or dial forms.

Nondegenerate transformations, if applied to identical relations, enable one to obtain new identical relations and new closed exterior differential forms.

The nondegenerate transformations proceed the transitions between the closed forms of different degrees. With such nondegenerate transformations of the exterior differential forms many operators of mathematical physics are connected. If, in addition to the exterior differential, we introduce the following operators: 1) δ for transformations that convert the form of $(p+1)$ degree into the form of p degree, 2) δ' for cotangent transformations, 3) Δ for the $d\delta-\delta d$ transformation, 4) Δ' for the $d\delta' - \delta' d$ transformations, one can see that the operator δ corresponds to Green's operator, δ' does to the canonical transformation operator, Δ does to the d'Alembert operator in 4-dimensional space, and Δ' corresponds to the Laplace operator [5].

Hence, one can see that many identical relations and nondegenerate transformations, which occur in various branches of mathematics, are connected with the properties of closed exterior (skew-symmetric) differential forms. This enables one to understand the properties and specific features of such identical

relations and nondegenerate transformations and to introduce their classification. (And yet this discloses potentialities of exterior differential forms.)

The role of identical relations in mathematics and physics consists in the fact that they define invariant and covariant objects, which are of great functional and utilitarian importance. The above considered identical relations, which correspond to closed exterior and dual forms, define the objects, which remain to be invariant and covariant under all transformations preserving the differential. I should be noted that an invariant object, to which the closed form corresponds, and the covariant object, to which the dual form corresponds, form a double structure being the example of the differential and geometric G-structure. In physics such structures describe the physical structures that forms physical fields and relevant manifolds [10,11].

2 Nonidentical relations and degenerate transformations

Identical relations and nondegenerate transformations, which describe invariant objects and transitions between invariant objects, cannot explain how do the invariant objects appear and what do them generate. To answer this question one must understand how do the identical relations realize. The theory of exterior differential forms, which is invariant one, cannot give the answer to such questions. To do this, the evolutionary theory is necessary. In the author's works it has been sown that there exist a skew-symmetric differential forms, which possess the evolutionary properties. They form nonidentical relations from which the identical relations, corresponding to invariant objects, are obtained by application of the degenerate transformations.

Evolutionary differential forms

Differential forms, which possess the evolutionary properties and so were called the evolutionary differential forms, appear from the description of various process.

A radical distinction between the evolutionary forms and the exterior ones consists in the fact that the exterior differential forms are defined on manifolds with *closed metric forms*, whereas the evolutionary differential forms are defined on manifolds with *unclosed metric forms*.

The evolutionary differential form of degree p (p-form), as well as the exterior differential form, can be written down as

$$\omega^p=\sum_{\alpha_1\dots\alpha_p}a_{\alpha_1\dots\alpha_p}dx^{\alpha_1}\wedge dx^{\alpha_2}\wedge\dots\wedge dx^{\alpha_p}\quad 0\leq p\leq n \tag{2.1}$$

But the evolutionary form differential cannot be written similarly to that presented for exterior differential forms (see formula (1.3)). In the evolutionary

form differential there appears an additional term connected with the fact that the basis of the form changes. For differential forms defined on the manifold with unclosed metric form one has $d(dx^{\alpha_1}dx^{\alpha_2}\dots dx^{\alpha_p})\neq 0$ (it should be noted that for differentiable manifold the following is valid: $d(dx^{\alpha_1}dx^{\alpha_2}\dots dx^{\alpha_p})=0$). For this reason the differential of the evolutionary form ω^p can be written as

$$d\omega^p=\sum_{\alpha_1\dots\alpha_p} da_{\alpha_1\dots\alpha_p}dx^{\alpha_1}dx^{\alpha_2}\dots dx^{\alpha_p}+\sum_{\alpha_1\dots\alpha_p} a_{\alpha_1\dots\alpha_p}d(dx^{\alpha_1}dx^{\alpha_2}\dots dx^{\alpha_p}) \tag{2.2}$$

where the second term is connected with the differential of the basis. The second term is expressed in terms of the metric form commutator. For the manifold with closed metric form this term vanishes.

For example, we again inspect the first-degree form $\omega=a_\alpha dx^\alpha$. The differential of this form can be written as $d\omega=K_{\alpha\beta}dx^\alpha dx^\beta$, where $K_{\alpha\beta}=a_{\beta;\alpha}-a_{\alpha;\beta}$ are components of the commutator of the form ω, and $a_{\beta;\alpha}$, $a_{\alpha;\beta}$ are the covariant derivatives. If we express the covariant derivatives in terms of the connectedness (if it is possible), they can be written as $a_{\beta;\alpha}=\partial a_\beta/\partial x^\alpha+\Gamma^\sigma_{\beta\alpha}a_\sigma$, where the first term results from differentiating the form coefficients, and the second term results from differentiating the basis. (In Euclidean space covariant derivatives coincide with ordinary ones since in this case derivatives of the basis vanish). If we substitute the expressions for covariant derivatives into the formula for the commutator components, we obtain the following expression for the commutator components of the evolutionary form ω:

$$K_{\alpha\beta}=\left(\frac{\partial a_\beta}{\partial x^\alpha}-\frac{\partial a_\alpha}{\partial x^\beta}\right)+(\Gamma^\sigma_{\beta\alpha}-\Gamma^\sigma_{\alpha\beta})a_\sigma \tag{2.3}$$

Here the expressions $(\Gamma^\sigma_{\beta\alpha}-\Gamma^\sigma_{\alpha\beta})$ entered into the second term are just the components of commutator of the first-degree metric form.

That is, the corresponding metric form commutator will enter into the differential form commutator.

The evolutionary properties of evolutionary differential forms are just connected with the properties of commutator of this form.

The evolutionary differential form commutator, in contrast to that of the exterior one, cannot be equal to zero because it involves the metric form commutator being nonzero. This means that the evolutionary form differential is nonzero. Hence, the evolutionary differential form, in contrast to the case of the exterior form, cannot be closed.

he commutators of evolutionary forms depend not only on the evolutionary form coefficients, but on the characteristics of manifolds, on which this form is defined, as well. As a result, such a dependence of the evolutionary form commutator produces the topological and evolutionary properties of both the commutator and the evolutionary form itself (this will be demonstrated below).

Since the evolutionary differential forms are unclosed, the mathematical apparatus of evolutionary differential forms does not seem to possess any pos-

sibilities connected with the algebraic, group, invariant and other properties of closed exterior differential forms. However, the mathematical apparatus of evolutionary forms includes some new unconventional elements.

Nonidentical relations of evolutionary differential forms

Above it was shown that the identical relations lie at the basis of the mathematical apparatus of exterior differential forms.

In contrast to this, nonidentical relations lie at the basis of the mathematical apparatus of evolutionary differential forms.

The identical relations of closed exterior differential forms reflect a conjugacy of any objects. The evolutionary forms, being unclosed, cannot directly describe a conjugacy of any objects. But they allow a description of the process in which the conjugacy may appear (the process when closed exterior differential forms are generated). Such a process is described by nonidentical relations.

The concept of "nonidentical relation" may appear to be inconsistent. However, it has a deep meaning.

The identical relations establish exact correspondence between the quantities (or objects) involved into this relation. This is possible in the case when the quantities involved into the relation are measurable ones. [A quantity is called a measurable quantity if its value does not change under transition to another, equivalent, coordinate system. In other words, this quantity is invariant one.] In the nonidentical relations one of the quantities is unmeasurable. (Nonidentical relations with two unmeasurable quantities are senseless). If this relation is evolutionary one, it turns out to be a selfvarying relation, namely, a variation of some object leads to variation of other one, and in turn a variation of the second object leads to variation of the first and so on. Since in the nonidentical relation one of the objects is a unmeasurable quantity, the other object cannot be compared with the first, and therefore, the process cannot stop. Here the specific feature is that in the process of such selfvarying it may be realized the additional conditions under which the identical relation can be obtained from nonidentical relation. The additional condition may be realized spontaneously while selfvarying the nonidentical relation if the system (which is described by such relation) possesses any symmetry. When such additional conditions are realized the exact correspondence between the quantities involved in the relation is established. Under the additional condition a unmeasurable quantity becomes a measurable quantity as well, and the exact correspondence between the objects involved in the relation is established. That is, an identical relation can be obtained from a nonidentical relation.

The nonidentical relation is a relation between a closed exterior differential form, which is a differential and is a measurable quantity, and an evolutionary form, which is an unmeasurable quantity.

Nonidentical relations of such type appear under descriptions of any pro-

cesses. These relations may be written as

$$d\psi = \omega^p \tag{2.4}$$

Here ω^p is the p-degree evolutionary form that is nonintegrable, ψ is some form of degree $(p-1)$, and the differential $d\psi$ is a closed form of degree p.

In the left-hand side of this relation it stands the form differential, i.e. the closed form, which is an invariant object. In the right-hand side it stands the nonintegrable unclosed form, which is not an invariant object. Such a relation cannot be identical.

One can see a difference of relations for exterior forms and evolutionary ones. In the right-hand side of identical relation (see relation (1.11)) it stands a closed form, whereas the form in the right-hand side of nonidentical relation (2.4) is an unclosed one.

Such nonidentical relations appear, for example, while investigating the integrability of any differential equations that describe various processes. The equation is integrable if it can be reduced to the form $d\psi = dU$. However it appears that, if the equation is not subject to an additional condition (the integrability condition), it is reduced to form (2.4), where ω is an unclosed form and it cannot be expressed as a differential. In the works [6,10] there are presented nonidentical relations obtained while investigating the integrability of the first-order partial differential equation and while analyzing the physical processes in material media. (it could be noted that the differential in the left-hand side of the nonidentical relation specifies the state of the medium or system described, and $d\psi$ can be a state function.)

The first principle of thermodynamics is an example of nonidentical relation [8].

It arises a question of how to work with nonidentical relation?

Two different approaches are possible.

The first, evident, approach is to find the condition under which the nonidentical relation becomes identical and to obtain a closed form under this condition. In other words, the nonidentical relation is subject to the condition, under which this relation is transformed into an identical relation (if it is possible).

Such an approach is traditional and is always used implicitly. It may be shown that additional conditions are imposed on the mathematical physics equations obtained in description of the physical processes so that these equations should be invariant (integrable) or should have invariant solutions.

This approach does not solve the evolutionary problem. {Here a psychological point should be noted. While investigating real physical processes one often faces the relations that are nonidentical. But it is commonly believed that only identical relations can have any physical meaning. For this reason one immediately attempts to impose a condition onto the nonidentical relation under which this relation becomes identical, and it is considered only such cases when this relation can satisfy the additional conditions. And all remaining is rejected. It

is not taken into account that a nonidentical relation is often obtained from a description of some physical process and it has physical meaning at every stage of the physical process rather than at the stage when the additional conditions are satisfied. In essence, the physical process does not considered completely. At this point it should be emphasized that the nonidentity of the evolutionary relation does not mean the imperfect accuracy of the mathematical description of a physical process. The nonidentical relations are indicative of specific features of the physical process development.}

The second approach to investigating the nonidentical evolutionary relation shows that the conditions, under which from the nonidentical relation it is obtained the identical relation, are realized under selfvariation of the nonidentical relation.

Selfvariation of the evolutionary nonidentical relation

The evolutionary nonidentical relation is selfvarying, because, firstly, it is nonidentical, namely, it contains two objects one of which appears to be unmeasurable, and, secondly, it is an evolutionary relation, namely, a variation of any object of the relation in some process leads to variation of another object and, in turn, a variation of the latter leads to variation of the former. Since one of the objects is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot stop.

Varying the evolutionary form coefficients leads to varying the first term of the evolutionary form commutator (see (2.3)). In accordance with this variation it varies the second term, that is, the metric form of manifold varies. Since the metric form commutators specifies the manifold differential characteristics, which are connected with the manifold deformation (as it has been pointed out, the commutator of the zero degree metric form specifies the bend, that of second degree specifies various types of rotation, that of the third degree specifies the curvature), this points to the manifold deformation. This means that it varies the evolutionary form basis. In turn, this leads to variation of the evolutionary form, and the process of intervariation of the evolutionary form and the basis is repeated. Processes of variation of the evolutionary form and the basis are controlled by the evolutionary form commutator and it is realized according to the evolutionary relation.

The process of the evolutionary relation selfvariation plays a governing role in description of the evolutionary processes.

The significance of the evolutionary relation selfvariation consists in the fact that in such a process it can be realized the conditions under which the identical relation is obtained from the nonidentical relation. These are the conditions of degenerate transformation.

Degenerate transformations.

To obtain the identical relation from the evolutionary nonidentical relation, it is necessary that a closed exterior differential form should be derived from the evolutionary differential form, which is included into evolutionary relation. However, as it has been shown above, the evolutionary form cannot be a closed form. For this reason a transition from the evolutionary form is possible only to an *inexact* closed exterior form, which is defined on pseudostructure.

To the pseudostructure it is assigned a closed dual form (whose differential vanishes). For this reason a transition from the evolutionary form to a closed inexact exterior form proceeds only when the conditions of vanishing the dual form differential are realized, in other words, when the metric form differential or commutator becomes equal to zero.

Since the evolutionary form differential is nonzero, whereas the closed exterior form differential is zero, the transition from the evolutionary form to the closed exterior form is allowed only under *degenerate transformation*. The conditions of vanishing the dual form differential (the additional condition) are the conditions of degenerate transformation.

Such conditions can just be realized under selfvariation of the nonidentical evolutionary relation.

As the conditions of degenerate transformation (additional conditions) it can serve any symmetries of the evolutionary form coefficients or of its commutator. (While describing material system such additional conditions are related, for example, to degrees of freedom of the material system).

Mathematically to the conditions of degenerate transformation there corresponds a requirement that some functional expressions become equal to zero. Such functional expressions are Jacobians, determinants, the Poisson brackets, residues, and others.

Obtaining identical relation from nonidentical one

Let us consider nonidentical evolutionary relation (2.4).

As it has been already mentioned, the evolutionary differential form ω^p, involved into this relation is an unclosed one. The commutator, and hence the differential, of this form is nonzero. That is,

$$d\omega^p \neq 0 \tag{2.5}$$

If the transformation is degenerate, from the unclosed evolutionary form it can be obtained a differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition

$d\omega^p \neq 0 \to$ (degenerate transformation) $\to d_\pi \omega^p = 0$, $d_\pi{}^*\omega^p = 0$

On the pseudostructure π evolutionary relation (2.4) transforms into the relation

$$d_\pi \psi = \omega_\pi^p \tag{2.6}$$

which proves to be the identical relation. Indeed, since the form ω_π^p is a closed one, on the pseudostructure this form turns out to be a differential of some differential form. In other words, this form can be written as $\omega_\pi^p = d_\pi\theta$. Relation (2.11) is now written as

$$d_\pi\psi = d_\pi\theta$$

There are differentials in the left-hand and right-hand sides of this relation. This means that the relation is an identical one.

From evolutionary relation (2.4) it is obtained the identical on the pseudostructure relation. In this case the evolutionary relation itself remains to be nonidentical one. (At this point it should be emphasized that differential, which equals zero, is an interior one. The evolutionary form commutator becomes zero only on the pseudostructure. The total evolutionary form commutator is nonzero. That is, under degenerate transformation the evolutionary form differential vanishes only *on pseudostructure*. The total differential of the evolutionary form is nonzero. The evolutionary form remains to be unclosed.)

It can be shown that all identical relations of the exterior differential form theory are obtained from nonidentical relations (that contain the evolutionary forms) by applying degenerate transformations.

The degenerate transform is realized as a transition to nonequivalent coordinate system: the transition from the accompanying noninertial coordinate system to the locally inertial that. Evolutionary relation (2.4) and condition (2.5) relate to the system being tied to the accompanying manifold, whereas identical relations (2.6) may relate only to the locally inertial coordinate system being tied to a pseudostructure.

Integration of the nonidentical evolutionary relation

Under degenerate transformation from the nonidentical evolutionary relation one obtains a relation being identical on pseudostructure. Since the right-hand side of such a relation can be expressed in terms of differential (as well as the left-hand side), one obtains a relation that can be integrated, and as a result he obtains a relation with the differential forms of less by one degree.

The relation obtained after integration proves to be nonidentical as well.

The resulting nonidentical relation of degree $(p-1)$ (relation that contains the forms of the degree $(p-1)$) can be integrated once again if the corresponding degenerate transformation has been realized and the identical relation has been formed.

By sequential integrating the evolutionary relation of degree p (in the case of realization of the corresponding degenerate transformations and forming the identical relation), one can get closed (on the pseudostructure) exterior forms of degree k, where k ranges from p to 0.

In this case one can see that under such integration the closed (on the pseudostructure) exterior forms, which depend on two parameters, are obtained.

These parameters are the degree of evolutionary form p (in the evolutionary relation) and the degree of created closed forms k.

In addition to these parameters, another parameter appears, namely, the dimension of space. If the evolutionary relation generates the closed forms of degrees $k = p$, $k = p - 1$, $\dots$, $k = 0$, to them there correspond the pseudostructures of dimensions $(N - k)$, where N is the space dimension. {It is known that to the closed exterior differential forms of degree k there correspond skew-symmetric tensors of rank k and to corresponding dual forms there do the pseudotensors of rank $(N - k)$, where N is the space dimensionality. The pseudostructures correspond to such tensors, but only on the space formed.}

Relation between degenerate and nondegenerate transformations

A peculiarity of the degenerate and nondegenerate transformations can be considered by the example of the field equation (the investigation of this equation is presented in the work [6]). In this case the degenerate transformation is a transition from the Lagrange function to the Hamilton function. The equation for the Lagrange function, that is the Euler variational equation, was obtained from the condition $\delta S = 0$, where S is the action functional. In the real case, when forces are nonpotential or couplings are nonholonomic, the quantity δS is not a closed form, that is, $d\,\delta S \neq 0$. But the Hamilton function is obtained from the condition $d\,\delta S = 0$ which is the closure condition for the form δS. The transition from the Lagrange function L to the Hamilton function H (the transition from variables q_j, $\dot{q}_j$ to variables q_j, $p_j = \partial L/\partial \dot{q}_j$) is a transition from the tangent space, where the form is unclosed, to the cotangent space with a closed form. One can see that this transition is a degenerate one.

The degenerate transformation is a transition from the tangent space $(q_j,\, \dot{q}_j))$ to the cotangent (characteristic) manifold $(q_j,\, p_j)$. On the other hand, the nondegenerate canonical transformation is a transition from one characteristic manifold $(q_j,\, p_j)$ to another characteristic manifold $(Q_j,\, P_j)$. {The formula of nondegenerate canonical transformation can be written as $p_j dq_j = P_j dQ_j + dW$, where W is the generating function}.

Here it has been shown a connection between the canonical nondegenerate transformation and the degenerate transformation. It may be easily shown that such a property of duality is also a specific feature of transformations such as tangent, gradient, contact, gauge, conform mapping, and others.

Transition from nonconjugated operators to conjugated operators

The evolutionary process of obtaining the identical relation from the nonidentical one and obtaining a closed (inexact) exterior form from the unclosed evolutionary form describes a process of conjugating any objects.

In section 1 it has been shown that the condition of the closure of exterior differential forms is a result of the conjugacy of any constituents of the exterior or dual forms (the form elements, components of each element, exterior and dual forms, exterior forms of various degrees, and others). Since the identical relations of exterior differential forms is a mathematical record of the closure conditions of exterior differential forms and, correspondingly, of conjugacy of any objects, the process of obtaining the identical relation from nonidentical one (selfmodification of the nonidentical evolutionary relation and degenerate transformation) is a process of conjugating any objects.

Here it could be pointed out the following. To the differential of the closed exterior differential form there correspond conjugated operators being equal to zero, whereas to the differential of the evolutionary form there correspond nonconjugated operators being not equal to zero. The transition from the evolutionary form to the closed exterior form is that from nonconjugated operators to conjugated ones. This is expressed mathematically as a transition from a nonzero differential (the evolutionary form differential is nonzero) to a differential that equals zero (the closed exterior form differential equals zero). his reveals as the transition from one coordinate system to another (nonequivalent) coordinate system.

Summary.

The mathematical apparatus of exterior and evolutionary skew-symmetric differential forms constitute a new closed mathematical apparatus that possesses the unique properties. It includes new, unconventional, elements: "nonidentical relation", "degenerate transformation", "transition from one frame of reference to another, nonequivalent, frame of reference". This enables one to create the mathematical language that has radically new abilities.

Identical and nonidentical relations, nondegenerate and degenerate transformations, transitions from nonidentical relations to identical ones are of great functional and utilitarian importance. First of all, this importance consists in the fact that, using the degenerate transformations, the nonidentical relations generate the identical relations and closed exterior forms, which describe invariant and covariant objects and the differential and geometrical structures. This discloses the mechanism of evolutionary processes, discrete transitions, generation of various structures.

The utilitarian importance of nonidentical relations and degenerate transformations consists in the fact that they disclose a mechanism of the evolutionary processes in material media and a process of creating physical structures and forming physical fields and manifolds. In more detail this is outlined in the works [10,11].

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